XXV. *On the Mysteries of Numbers alluded to by* FERMAT.—Second Communication.

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IN the last paper I described the mode of constructing a square (which, for brevity, I shall call *The Square*) which would necessarily be made up of three different series; the indices in the margin and in the small squares were explained, and I must refer to that paper for the explanations.

I propose in this paper to show (from *that square* and a *supplemental one*) that all the odd numbers possess the properties that I have ascribed to them. *The Square* also proves the first theorem of FERMAT, viz. that every number is composed of three triangular numbers or less from the 2nd theorem (relating to the squares), which I believe has not hitherto been done.

If any one will take the trouble to examine any odd number, he will find that it may be divided into four squares (generally in several ways or forms) in some set of four roots of the squares composing the number; two of the roots will be equal, two of the same, or another set of four will differ by 1; two will differ by 2, two by 3, and so on, as far as the number is large enough to have roots of sufficient magnitude to furnish such differences. This, of course, cannot be done by one set of roots, except by a few of the low numbers*. If the four roots do not furnish all the differences, then he will discover that there is another or more forms that will furnish the whole of them.

The algebraic sum of the four roots will also, in some or other of the forms, be equal to 1, 3, 5, 7, &c., that is, to every odd number within the compass of the given number. These properties, which may be discovered in any odd number, I propose to show belong to all odd numbers.

It may be well, in order to make this statement quite clear, to take an example. 73 is one of the terms in the series 1, 3, 7, 13, 21, &c., a series increasing by 2, 4, 6, 8, &c.; every term of the series has roots of the form $n, n, n, n\pm 1$.

73 is composed of four squares in four different forms; the four sets of roots are

		Differences of roots.
4, 4, 4, 5	. . . .	0, 1, 8, 9
2, 2, 4, 7	. . . .	2, 3, 4, 5, 6, 11
0, 1, 6, 6	. . . .	5, 7, 12, &c.
1, 2, 2, 8	. . . .	10, &c.

* 3, 5, 7, 11, 15, and 23 have each but one form, and the last (23) is the highest odd number that has only one form of roots.

These forms give the differences of roots set opposite to them; 12 is the limit, as the lowest number which can give a difference of 13 is 85 (0, 0, 6, 7). 6.7 have a difference of 13.
roots.

So with reference to the algebraic sum of the roots:

4, 4, 4, 5 gives 1, 7, 9, and 17
2, 2, 4, 7 gives 1, 3, 7, 11, 15
0, 1, 6, 6 gives 1, 11, 13
1, 2, 2, 8 gives 3, 5, 7, 9, 11, 13.

It is remarkable that the sum of the roots being equal to every odd number is immediately connected with FERMAT'S first proposition of the triangular numbers; it is also remarkable that the second proposition of the differences of the roots being equal to every number, odd or even, is immediately connected with FERMAT'S second proposition of every number consisting of four squares or less. The connexion between the sum of the roots and FERMAT'S first proposition was observed by me in the year 1854, and is mentioned in a paper which the Royal Society did me the honour to publish in the Philosophical Transactions for that year, vol. cxliv. p. 315, as Theorem C: see also p. 318.

Before I proceed to show the other properties of *The Square*, I think it right to call attention to the manner in which a change in the roots of the four squares alters the sum of the squares themselves. This subject was touched upon by me in a paper published in the Philosophical Transactions for 1859, vol. cxlix. p. 49, in which it is stated, and a sort of proof (not a satisfactory one) given, that in some form of division into four squares the roots will be equal, will differ by 1, by 2, by 3, &c.

I propose now to call attention to the effect, or result, of altering the roots so as to increase or decrease the sum of the squares. If the roots of two of the four squares that compose any odd number differ by n, and the larger of the two be increased by 1, and the smaller be decreased by 1, the sum of the squares will be increased by $2n+2$. Let p and $p+n$ be the roots that differ by n, the sum of their squares will be

$$2p^2+2pn+n^2.$$

If $p-1$ and $(p+n)+1$ be squared, the sum of their squares will be

$$p^2-2p+1+p^2+2pn+n^2+2p+2n+1,$$

or

$$2p^2+2pn+n^2+2n+2;$$

the increase is $2n+2$.

If $n=0$ the increase is 2,
if $n=1$ the increase is 4,
if $n=2$ the increase is 6,

and so on, always *twice the difference* $+2$.

There is a similar theorem for reducing the sum of the squares; if p and $p+n$ become

$p+1$ and $p+n-1$, the sum of the roots will then be diminished by $2n-2$. The algebraic sum of the roots is not altered by this increase of one and decrease of another of them by the same number.

For, if

$$\pm a \pm b \pm c \pm d = 1, \text{ or } 2n+1,$$

then

$$\pm a \pm (b \mp n) \pm (c \pm n) \pm d = 1, \text{ or } 2n+1.$$

All these changes reciprocate; thus, if a number has two roots equal, the number which is greater by 2 will have two roots differing by 2; and *vice versâ*, if a number has two roots differing by 2, the number which is less by 2 will have two roots equal; thus, if 13 has two roots differing by 2 as 0, 2, then 11, which is less by 2, will have two roots equal, viz. 1, 1.

If an odd number has roots differing by 0, 1, 2, 3, 4, 5, &c., the four roots of the odd numbers, next in succession, one after another, will be discovered till the differences are exhausted; and if *every* number has this property, the succession will continue through the whole series of odd numbers, and every odd number will be composed of four squares, and therefore every number will be so likewise; for every even number is ultimately the double of an odd number, and the double of four squares is itself composed of four squares.

There is another mode of altering the roots, substantially the same, but occasionally applicable when the other is not. It is mentioned as Theorem A in page 313 of the Philosophical Transactions for 1854; but I was not *then* aware of its full effect, and did not pursue the subject.

If the difference between the sum of two of the four roots, and the sum of the other two be n, and each of the larger be decreased by n, and each of the smaller e bincreased by n, the increase in the sum of the squares will be $2n$, if $n=1$ the increase is 2.

This is a proper place to mention a property of the series mentioned above, 1, 3, 7, 13, &c. In a former paper (see Philosophical Transactions, 1854, page 315) I called this a *gradation series*, but I was not then familiar with its properties, and did not pursue the subject as I propose now to do. Every term in the series has roots of the form $n, n, n, n\pm 1$. If any term, as 13 (which is composed of the squares of the four roots 1, 2, 2, 2), be increased by the methods above mentioned, taking care to keep the sum of the roots always equal to the number 7 (the sum of the roots of 13), it will be found that, from 13 to 21 inclusive, every odd number will be composed of four squares, the sum of whose roots is 7. In this case the process cannot be carried beyond 21; for 23 being of the form $8n+7$, cannot be composed of less than four squares: 5 is too large to be one of the roots, and four is unavoidable, since $23-16$ leaves a remainder of 7, which cannot be expressed by less than four squares. The only form, therefore, of roots for 23 is 1, 2, 3, 3, and their sum is 9. But the sum of the roots from the number 21 to 31 will be 9; and from 21 to 31 the odd numbers can be made of four roots whose sum shall in every case be 9; and this goes two steps beyond 31; and as the number

increases which may belong to the sum of the roots, the number of odd numbers in succession which may be formed increases faster than the interval between the terms of the *gradation series.* This will appear by an examination of the numbers themselves; but the interval may be filled up more readily in the manner above mentioned; and if any two terms in the *gradation series* be taken, the odd numbers from the one to the other, both inclusive, may be made up of the squares of roots whose sum shall be the number indicated by the sum of the roots of the first of the two terms.

Thus 3, 3, 3, 4 are the roots whose squares make the number 43; and from 43 to 57, which is composed of 3, 4, 4, 4, every odd number can be found, the sum of whose roots shall equal 13. Thus

From 43 to 57 is but 7 steps; but the series of odd numbers consisting of squares whose roots equal 13 goes on 6 steps further.

$$\left.\begin{array}{l}
3 \overset{0}{,} 3 \overset{0}{,} 3 \overset{1}{,} 4 = 43 \\
2 \overset{1}{,} 3 \overset{1}{,} 4 \overset{0}{,} 4 = 45 \\
2 \overset{1}{,} 3 \overset{0}{,} 3 \overset{2}{,} 5 = 47 \\
2 \overset{0}{,} 2 \overset{2}{,} 4 \overset{1}{,} 5 = 49 \\
1 \overset{2}{,} 3 \overset{1}{,} 4 \overset{1}{,} 5 = 51 \\
2 \overset{0}{,} 2 \overset{1}{,} 3 \overset{3}{,} 6 = 53 \\
1 \overset{2}{,} 3 \overset{0}{,} 3 \overset{3}{,} 6 = 55 \\
1 \overset{1}{,} 2 \overset{2}{,} 4 \overset{2}{,} 6 = 57 (= 3, 4, 4, 4)
\end{array}\right\}$$

$$\begin{array}{l}
0 \overset{3}{,} 3 \overset{2}{,} 5 \overset{0}{,} 5 = 59 \\
0 \overset{3}{,} 3 \overset{1}{,} 4 \overset{2}{,} 6 = 61 \\
1 \overset{1}{,} 2 \overset{1}{,} 3 \overset{4}{,} 7 = 63 \\
0 \overset{2}{,} 2 \overset{3}{,} 5 \overset{1}{,} 6 = 65 \\
1 \overset{0}{,} 1 \overset{3}{,} 4 \overset{3}{,} 7 = 67 \\
0 \overset{2}{,} 2 \overset{2}{,} 4 \overset{3}{,} 7 = 69
\end{array}$$

The differences between the roots are marked in order to compare them with those in the next number

So

From 57 to 73 is 8 steps (one more than the last); but the series of odd numbers goes on 10 steps further.

$$\left\{\begin{array}{l}
3 \overset{1}{,} 4 \overset{0}{,} 4 \overset{0}{,} 4 = 57 \\
3 \overset{0}{,} 3 \overset{1}{,} 4 \overset{1}{,} 5 = 59 \\
2 \overset{2}{,} 4 \overset{0}{,} 4 \overset{1}{,} 5 = 61 \\
2 \overset{1}{,} 3 \overset{2}{,} 5 \overset{0}{,} 5 = 63 \\
2 \overset{1}{,} 3 \overset{1}{,} 4 \overset{2}{,} 6 = 65 \\
1 \overset{3}{,} 4 \overset{1}{,} 5 \overset{0}{,} 5 = 67 \\
1 \overset{3}{,} 4 \overset{0}{,} 4 \overset{2}{,} 6 = 69 \\
1 \overset{2}{,} 3 \overset{2}{,} 5 \overset{1}{,} 6 = 71 \\
2 \overset{0}{,} 2 \overset{2}{,} 4 \overset{3}{,} 7 = 73 = 4, 4, 4, 5
\end{array}\right.$$

$$\begin{array}{l}
1 \overset{2}{,} 3 \overset{1}{,} 4 \overset{3}{,} 7 = 75 \\
0 \overset{4}{,} 4 \overset{1}{,} 5 \overset{1}{,} 6 = 77 \\
1 \overset{1}{,} 2 \overset{3}{,} 5 \overset{2}{,} 7 = 79 \\
0 \overset{3}{,} 3 \overset{3}{,} 6 \overset{0}{,} 6 = 81 \\
0 \overset{3}{,} 3 \overset{2}{,} 5 \overset{2}{,} 7 = 83 \\
1 \overset{1}{,} 2 \overset{2}{,} 4 \overset{4}{,} 8 = 85 \\
1 \overset{0}{,} 1 \overset{5}{,} 6 \overset{1}{,} 7 = 87 \\
0 \overset{2}{,} 2 \overset{4}{,} 6 \overset{1}{,} 7 = 89 \\
1 \overset{0}{,} 1 \overset{4}{,} 5 \overset{3}{,} 8 = 91 \\
0 \overset{2}{,} 2 \overset{3}{,} 5 \overset{3}{,} 8 = 93
\end{array}$$

The differences are the same, but are reversed, and this occurs alternately.

Again:

Here the next term in the gradation series (91) is reached by 9 steps; but the series of odd numbers goes on 15 steps further.

4, 4, 4, 5 = 73
3, 4, 5, 5 = 75
3, 4, 4, 6 = 77
3, 3, 5, 6 = 79
2, 4, 5, 6 = 81
3, 3, 4, 7 = 83
2, 4, 4, 7 = 85
2, 3, 5, 7 = 87
1, 4, 6, 6 = 89
1, 4, 5, 7 = 91 = 4, 5, 5, 5
2, 3, 4, 8 = 93
1, 3, 6, 7 = 95
2, 2, 5, 8 = 97
1, 3, 5, 8 = 99
0, 4, 6, 7 = 101
1, 2, 7, 7 = 103
1, 2, 6, 8 = 105
0, 3, 7, 7 = 107

3 3 2
0, 3, 6, 8 = 109

1 3 4
1, 2, 5, 9 = 111

4 0 5
0, 4, 4, 9 = 113

3 2 4
0, 3, 5, 9 = 115

2 5 3
0, 2, 7, 8 = 117

0 5 3
1, 1, 6, 9 = 119

2 4 3
0, 2, 6, 9 = 121

4, 5, 5, 5 =	91
4, 4, 5, 6 =	93
3, 5, 5, 6 =	95
3, 4, 6, 6 =	97
3, 4, 5, 7 =	99
2, 5, 6, 6 =	101
2, 5, 5, 7 =	103
2, 4, 6, 7 =	105
3, 3, 5, 8 =	107
2, 4, 5, 8 =	109
1, 5, 6, 7 =	111 = 5, 5, 5, 6
2, 3, 6, 8 =	113
1, 4, 7, 7 =	115
1, 4, 6, 8 =	117
2, 3, 5, 9 =	119
2, 2, 7, 8 =	121
1, 3, 7, 8 =	123
2, 2, 6, 9 =	125
1, 3, 6, 9 =	127
0, 4, 7, 8 =	129
0, 5, 5, 9 =	131
0, 4, 6, 9 =	133
1, 2, 7, 9 =	135
0, 3, 8, 8 =	137
0, 3, 7, 9 =	139
1, 2, 6, 10 =	141.

So much for the series 1, 3, 7, 13, 21, &c.; but *The Square* has three series. The other two (when the first term is 1) are 1, 5, 13, 25, 41, 61, &c., and 1, 3, 9, 19, 33, 51, 71, &c.; and each of these affords a similar facility of passing from one term to the next; but these last series assist each other, and the condition is, not the sum of all the roots being the same, but the difference of two of them, and any term of the second being halfway between two of the first; thus $25+8=33$, and $33+8=41$, which is true throughout both series.

	Roots.
25 =	3, 0, 0, 4
27 =	3, 1, 1, 4
29 =	3, 0, 2, 4
31 =	2, 1, 1, 5
33 =	2, 0, 2, 5

But 33 is in the other series, and is represented by

$$4, 0, 1, 4 = 33$$
$$3, 0, 1, 5 = 35$$
$$4, 1, 2, 4 = 37$$
$$3, 1, 2, 5 = 39$$
$$4, 0, 3, 4 = 41$$

The first five terms have each two roots whose difference is 7; the second five terms have each two roots whose difference is 8; and this process may be continued throughout the two series. 51 is the middle term between 41 and 61; 73 is the middle term between 61 and 85; and each of these admits of the same treatment as 25, 33, and 41.

If a *Supplemental Square* be appended to *The Square*, and be constructed on the reverse principle of diminishing towards the *left* as the other increases towards the right, beginning with precisely the same odd number (see Diagram No. 1), it is manifest that a series of numbers may be obtained less than the original number, and which will finally terminate, giving all the numbers from which the odd number in the beginning of *The Square* may proceed, if each be placed in succession in the first position of *The Square*; for it is clear that if the given odd number be lessened by 4, 8, 12, &c., and each of these numbers be again lessened by 2, 6, 10, &c., till the operation can be carried no further, the process which takes place in the formation of *The Square* itself will be reversed, and therefore if any number so obtained in *The Supplemental Square* be placed in the first position in *The Square*, and 4, 8, 12, &c. be added, and then to each of these 2, 6, 10, &c., at length the given odd number will be reached; and in this way it will be seen that the odd number will be found in any position whatever that its magnitude (as a number) would enable it to fill and properly occupy. And now, if all the numbers less than the given number have the properties, which it is alleged belong to all odd numbers, viz. of differences of two roots and sum of all the roots, then the given odd number will be accompanied by numbers less than itself, having such properties that they properly (that is, according *to the laws of The Square*) occupy their place in *The Square*; and it must follow that their composition will indicate what roots the given odd number ought to have in order to comply with the exigencies of the place in *The Square* that it occupies; for, in respect to any one of the series that compose *The Square*, if one

term in the series be correct, that is, have the proper sum of two roots, or the proper sum of all the roots, then every other term in that series will also be correct (see last paper) and may be derived from that term, and it will therefore have all the qualities that belong to the different portions of *The Square*, and in every one it will be divided into four squares, and the roots will appear. It may be said the first does not appear to have two roots equal, or two differing by 1, but (see Diagram No. 1) where the top row is called A, the second B, the third C, the fourth D, and the terms in each row are distinguished by 1, 2, 3, 4, and because 25 in A 1 has two roots differing by 1, 29 in A 2 will have two roots differing by 3; and because 17 in A 1 has two roots differing by 1, 29 in A 3 will have two roots differing by 5; and because 5 has two roots differing by 1, 29 in A 4 will have two roots differing by 7. But in A 2, 29 (which is derived from 27 the sum of whose roots is 1) will have the sum of its roots 3; and for a similar reason in A 3 it will have the sum of its roots 5 (derived from 23 the sum of whose roots is 1 in B 2), and in A 4 it will have the sum of its roots 7.

But if the sum of the four roots $=2n+1$ and the difference between two of them $=2n+1$, the other two must be equal to one another, or they would prevent the sum of the roots being $2n+1$.

A similar mode of reasoning applies to show that 29 must have two of its roots differing by 1. For, because 27 has two roots equal, therefore 29 in B 1 (derived from 27) has two roots differing by 2, and in C 1 (derived from 21) has two roots differing by 4, and in D 1 (derived from 11) has two roots differing by 6. But if any number has the sum of its roots equal to $2n+1$ and to $2n+3$, and two of the roots differ by $2n+2$, the other two must differ by 1, or the sum of all the roots could not be both $2n+1$ and $2n+3$.

I propose now to point out some of the results of what has been already stated.

There can be no doubt that every number, odd or even, is composed of four squares or less. And whether this be proved by the *gradation series*, or by the combination of *The Square* and the *Supplemental Square* (if these furnish a proof), or by LAGRANGE's method from the prime numbers, or by any other, the result is the same, the *proposition is true*; and it follows that a number of the form $4n+2$ must be composed of two odd squares, or of two odd squares and one even one, or two odd squares and two even ones.

In the first case $4n+2=4a^2+4a+1+4b^2+4b+1$; deducting 2 from each side and dividing by 4, $n=a^2+a+b^2+b$; that is, four trigonal numbers, of which two are equal and the other two are equal. In the second case $4n+2=4a^2+4a+1+4b^2+4b+1+4c^2$, $n=a^2+a+b^2+b+c^2$; but a^2+a+c^2 equals 2 trigonal numbers, therefore n equals four and trigonal numbers, of which two are equal. In the third case $n=a^2+a+b^2+b+c^2+d^2$; and as a^2+a+c^2 and b^2+b+d^2 are each equal to 2 trigonal numbers, n in every case is equal to four trigonal numbers, that is, to $a^2+a+b^2+c^2+c+d^2$.

In EULER's 'Opuscula Analytica,' vol. ii. p. 4, he says, "LAGRANGE's demonstration as to the square numbers is deduced from principles of such a kind that clearly no assistance can be expected from *that* to prove the rest of FERMAT's theorems*.

* Quæ autem ex ejusmodi principiis est deducta, ut inde nullum plane subsidium ad reliqua demonstranda expectari possit.

Now the above observation on a number of the form $4n+2$ shows that LAGRANGE'S method may be used to prove that every number is composed of four trigonal numbers as well as of four squares, and thus is brought into immediate connexion with *The Square*, and enables it to divide the first term (whatever odd number it may be) into 4 squares the algebraic sum of whose roots shall be 1, and the consequence of that is, that the $\frac{\text{1st term}-1}{2}$ will be a number composed of 3 trigonal numbers or less; it may be as well here to insert the proof of this.

If the sum of two of the roots differ from the sum of the other two by 1, the 2 sums of the roots must be of the form $2a+1$ and $2a$, and the four roots will be of the form $a+p+1$, $a-p$, $a+q$, $a-q$, and the sum of the roots squared will be

$$4a^2+2p^2+2q^2+2a+2p+1;$$

deducting 1, and dividing by 2, the number will be $2a^2+a+p^2+p+q^2$; $2a^2+a$ is a trigonal number, and p^2+p+q^2 is an expression for 2 trigonal numbers. Now if 1 be the first number in *The Square*, every number in it will be of the form $1+2a^2+2a+2b^2$, which is 4 squares, and expressed by their roots is $a+1$, a, b, b; and the sum of any two terms added together will be an even number; and as every possible value of a and b is to be found in *The Square*, every even number may be obtained by adding together some two of the numbers in *The Square*.

Another result is that, as every number is of the form $a^2+a+b^2+c^2+c+d^2$, every even number may be composed of 4 of the terms of the series 0, 2, 4, 8, 12, 18, 24, &c., the terms of which are alternately $2b^2$ and $2a^2+2a$.

The law of the series is obvious enough; beginning with 0, the differences are 2, 2, 4, 4, 6, 6, ... $2n$, $2n$. It is a convenient mode of using this series to place the terms in two columns, putting all the $2b^2$ in one column and all the $2a^2+2a$ in another, as below.

2	4
8	12
18	24
32	40
50	60
72	84
&c.	&c.

Every even number may be made by some 2 terms of each column, and any 2 squares, if equal, may be increased by 2, 8, 18, 32, &c., that is, by $2b^2$, by changing the roots—

		Increase.
n, n into	$n-1$, $n+1$	2
	$n-2$, $n+2$	8
	$n-3$, $n+3$	18
	&c. &c.	$2b^2$

In like manner any 2 roots that differ by 1 may be increased by 4, 12, 24, 40, &c. $(2a^2+2a)$ by changing

		Increase.
$n, n+1$ into	$n-1, n+2$	4
	$n-2, n+3$	12
	$n-3, n+4$	24
	&c. &c.	$(2a^2+2a)$

Another conclusion is, that if *The Square* have as the first number in it all the odd numbers *in succession* which are to be found in *The Square* when 1 is the first number, then every other odd number will be obtained as some one of the numbers thus formed. For every such *Square* will begin with a number of the form of $1+2a^2+2a+2b^2$; and in forming *The Square* from that, $2m^2+2m+2n^2$ will be added, therefore all the possible combinations of $2a^2+2a+2b^2$ and $2m^2+2m+2n^2$ will be found, with every value of a and b, m and n; that is, *every odd number will be found.*

But if any odd number whatever be made the first number in *The Square*, and a *Supplemental Square* be formed, and the numbers in the *Supplemental Square* be successively put in the first place in *The Square*, the assumed number will be found in some of the terms at the top, and also in some of those at the side. And the necessary consequence is, if we are allowed to notice the numbers of *The Square* when 1 is the first number, that any odd number is not only equal to $1+2a^2+2a+2b^2+2c^2+2c+2d^2$, but also to $1+2a^2+2a+2b^2+2c^2+2c$, or $1+2a^2+2a+2b^2+2d^2$; that is, the four squares may have two equal roots, or two roots differing by 1.

I now propose to show in what manner *The Square* can obtain a division of its first term into four squares, the algebraic sum of whose roots $=1$; the result of which is that the first term less 1, divided by 2, will be composed of 3 trigonal numbers (see Theorem C in Philosophical Transactions, vol. cxliv. p. 315).

Every odd number is of the form $4n+1$, or $4n+3$. If it be of the form $4n+3$, then in the $(2n+1)$th term of the series which increases by 2, 4, 6, 8, 10, &c., the roots will be n, $n+1$, $n+1$, $n+1$.

For $4n+3+2n$ terms of 2, 4, 6, 8, &c.

$$=4n+3+(2+4n)\times\frac{2n}{2}$$

$$=4n^2+6n+3=\begin{cases} n^2 \\ (n+1)^2 \\ (n+1)^2 \\ (n+1)^2 \end{cases}$$

the roots of which are n, $n+1$, $n+1$, $n+1$. The index of this term is $4n+1$ (since it is the $(2n+1)$th term of 1, 3, 5, 7, &c.), that is, 2 less than the sum of the roots.

In like manner, if the number be of the form $4n+1$, then the 2nth term in the series already mentioned will have as roots n, n, n, $n+1$.

For $4n+1+(2n-1)$ terms of 2, 4, 6, 8, &c.

$$=4n^2+2n+1=\begin{cases} n^2 \\ n^2 \\ n^2 \\ (n+1)^2 \end{cases}$$

which, when expressed in roots, is n, n, n, $n+1$. The index of the term is $4n-1$ (being the $2n$th term of 1, 3, 5, 7, &c.).

It will be seen that in each case the sum of the roots exceeds the index of the term by 2; it is therefore obvious that if the sum of the roots could be diminished by 2 (without altering the sum of the squares), the sum of the roots would be equal to the index, and the roots of every term in the diagonal would be obtained, with the sum of roots equal to the index of the term; and as the first number in *The Square* is in this diagonal, the roots of the given number would be obtained, whose sum would equal 1.

The difference between the sums of the squares of the roots of any two terms in the *gradation series* is the sum of the roots of the larger, minus 1. The difference between the sums of their roots is 2; therefore, if the sum of the roots of the larger, minus 1, can be added to the roots of the smaller, a set of roots will be obtained whose sum will be 2 less than that of the larger, but the sum of their squares will be equal to it. (See Diagram No. 2.)

Take the case of 35 as the first term of *The Square*. In the $(n+1)$th term of the diagonal, 8, 9, 9, 9 will be the roots of the squares which compose that term. The next term in the *gradation series* will have the roots 9, 9, 9, 10, but will be two more than the position requires, as will appear from calculation. The difference between $8^2+9^2+9^2+9^2$ and $9^2+9^2+9^2+10^2$ will be $9+9+9+10-1$, that is, 36, but that would be 2 too much. If, therefore, the squares of 8, 9, 9, 9 be increased by 34, the number will be what that place in *The Square* requires, and the sum of the roots will be 35, which is the index of that term.

Now the squares of the roots 8, 9, 9, 9 may be increased by 34 by adding $12+18+4$, which are three terms in the two columns above mentioned; 12 may be added by changing 8, 9 into 6, 11; and 18 may be added by changing 9, 9 into 6, 12. The roots then are 6, 11, 6, 12, and to these 4 is to be added. This may be done by changing 11, 12 into 10, 13, and the result is $6, \overset{0}{6}, \overset{4}{10}, \overset{3}{13}$, whose sum is 35, and is therefore equal to the index of the term.

To reduce roots whose sum is 35 to roots having the same differences but whose sum will equal 1, deduct 9 from each root, and 9×4 will be deducted, and the sum will then be -1, which by changing all the signs will become $+1$. From each of the roots 6, 6, 10, 13 deduct 9, and they will then be -3, -3, 1, 4, and the sum of their squares $=35$.

If 1 be deducted from 35, and the remainder be divided by 2, the quotient (17) will be composed of not exceeding three trigonal numbers.

There remains one other matter to be mentioned, viz. a certain remarkable relation which all the polygonal numbers bear to each other, and which forms a connexion that runs through them all; from which it would seem to follow that a solution of the theorem as to one, would be a solution as to all the rest (except the first).

This relation arises in the square numbers by a property of the gradation series, already in part alluded to, viz. as to the odd numbers, by which the interval between any two terms can be filled up, all the terms having, as to the odd numbers, the sum of the roots of the squares that compose them equal to the sum of the roots of the first term; but the intervals, as to the *even* numbers, may be also filled up by making the sum of the roots 1 less than that of the roots of the odd numbers (see the Table in Diagram No. 3), which is thus constructed: a term in the gradation series is assumed (in this case 73); its roots are 4, 4, 4, 5, and the roots of all the odd numbers between that and the next term are found by the processes mentioned in the former part of this paper. The roots of the even numbers are obtained by an analogous process, and these are used as bases or roots of the polygonal numbers, which are placed in columns, with their sums, as appears in the Table. See Diagram No. 4 for the mode in which the polygonal numbers are formed.

It will be observed that the sum of the roots or bases is 17, but if they be used to form trigonal numbers, the increment of the sum of the resulting trigonal numbers, above the sum of the roots or bases, is 28; and so on of the rest, each successive column increasing by the same number, viz. 28. If the roots or bases be n, n, n, $n \mp 1$, that is, a term in the gradation series, the increment of the sums of the successive columns will be $2n^2 \mp n$, a trigonal number.

Again, in the trigonal numbers the difference between the sums of the first and second term is 0; in the square numbers it is 1, in the pentagonal numbers 2, in the hexagonal numbers 3, in the heptagonal numbers 4; but in all of them the difference between the second and third terms is 1, and this continues throughout. The difference between the 3rd and 4th, the 5th and 6th, the 7th and 8th, &c. increases by 1 in each column, but the difference between the 2nd and 3rd, the 4th and 5th, the 6th and 7th, &c. is always 1 in each column; and the result is, that by adding 1 in the pentagonal column, by adding 1, or 1, 1 in the hexagonal, by adding 1, or 1, 1, or 1, 1, 1 in the heptagonal, every number, odd or even, can be made by not exceeding four square numbers, or five pentagonal numbers, or &c., as clearly appears by the Table.

This corresponds with what was discovered by CAUCHY, published at the end of LEGENDRE'S "Théorie des Nombres," viz. that four only of each class of numbers is necessary, the rest may be supplied by 1, repeated as often as necessary. But I must not omit to say that, although all the odd numbers are sufficiently obedient, there is one class of even numbers quite refractory, viz. the powers of 2. They may easily be expressed in squares, pentagonal numbers, &c., but they cannot be brought within the rule that otherwise prevails.

DIAGRAM No. 1.

					1	3	5	7	9
5	17	25	29	0 A	1 29	3 2	5 3	7 4	9
3	15	23	27	2 B	1 3 1	1 5 2	3 7 3	5 9 4	7 11
	9	17	21	4 C	3 5 1	1 7 2	1 9 3	3 11 4	5 13
		7	11	6 D	5 7 1	3 9 2	1 11	1 13	3 15
				8	7 9	5 11	3 13	1 15	1 17

Roots or Bases.					Sums.	Trigonal Numbers.					Sums.	Square Numbers.					Su
4	4	4	5	=	17	10	10	10	15	=	45	16	16	16	25	=	7
2	3	5	6	=	16	3	6	15	21	=	45	4	9	25	36	=	7
3	4	5	5	=	17	6	10	15	15	=	46	9	16	25	25	=	7
3	3	3	7	=	16	6	6	6	28	=	46	9	9	9	49	=	7
3	4	4	6	=	17	6	10	10	21	=	47	9	16	16	36	=	7
2	3	4	7	=	16	3	6	10	28	=	47	4	9	16	49	=	7
3	3	5	6	=	17	6	6	15	21	=	48	9	9	25	36	=	7
2	2	6	6	=	16	3	3	21	21	=	48	4	4	36	36	=	8
2	4	5	6	=	17	3	10	15	21	=	49	4	16	25	36	=	8
2	2	5	7	=	16	3	3	15	28	=	49	4	4	25	49	=	8
3	3	4	7	=	17	6	6	10	28	=	50	9	9	16	49	=	8
1	3	5	7	=	16	1	6	15	28	=	50	1	9	25	49	=	8
2	4	4	7	=	17	3	10	10	28	=	51	4	16	16	49	=	8
2	3	3	8	=	16	3	6	6	36	=	51	4	9	9	64	=	8
2	3	5	7	=	17	3	6	15	28	=	52	4	9	25	49	=	8
2	2	4	8	=	16	3	3	10	36	=	52	4	4	16	64	=	8
1	4	6	6	=	17	1	10	21	21	=	53	1	16	36	36	=	8
1	3	4	8	=	16	1	6	10	36	=	53	1	9	16	64	=	9
1	4	5	7	=	17	1	10	15	28	=	54	1	16	25	49	=	9
4	5	5	5	=	19	10	15	15	15	=	55	16	25	25	25	=	9
3	3	5	7	=	18	6	6	15	28	=	55	9	9	25	49	=	9

Diagram No 3.

Sums.	Square Numbers.				Sums.	Pentagonal Numbers.				Sums.	Hexagonal Numbers.		
= 45	16	16	16	25	= 73	22	22	22	35	= 101	28	28	28
= 45	4	9	25	36	= 74	5	12	35	51	= 103	6	15	45
= 46	9	16	25	25	= 75	12	22	35	35	= 104	15	28	45
= 46	9	9	9	49	= 76	12	12	12	70	= 106	15	15	15
= 47	9	16	16	36	= 77	12	22	22	51	= 107	15	28	28
= 47	4	9	16	49	= 78	5	12	22	70	= 109	6	15	28
= 48	9	9	25	36	= 79	12	12	35	51	= 110	15	15	45
= 48	4	4	36	36	= 80	5	5	51	51	= 112	6	6	66
= 49	4	16	25	36	= 81	5	22	35	51	= 113	6	28	45
= 49	4	4	25	49	= 82	5	5	35	70	= 115	6	6	45
= 50	9	9	16	49	= 83	12	12	22	70	= 116	15	15	28
= 50	1	9	25	49	= 84	1	12	35	70	= 118	1	15	45
= 51	4	16	16	49	= 85	5	22	22	70	= 119	6	28	28
= 51	4	9	9	64	= 86	5	12	12	92	= 121	6	15	15
= 52	4	9	25	49	= 87	5	12	35	70	= 122	6	15	45
= 52	4	4	16	64	= 88	5	5	22	92	= 124	6	6	28
= 53	1	16	36	36	= 89	1	22	51	51	= 125	1	28	66
= 53	1	9	16	64	= 90	1	12	22	92	= 127	1	15	28
= 54	1	16	25	49	= 91	1	22	35	70	= 128	1	28	45
= 55	16	25	25	25	= 91	22	35	35	35	= 127	28	45	45
= 55	9	9	25	49	= 92	12	12	35	70	= 129	15	15	45

gonal bers.		Sums.	Hexagonal Numbers.				Sums.	Heptagonal Numbers.				Sums.
22	35	= 101	28	28	28	45	= 129	34	34	34	55	= 157
35	51	= 103	6	15	45	66	= 132	7	18	55	81	= 161
35	35	= 104	15	28	45	45	= 133	18	34	55	55	= 162
12	70	= 106	15	15	15	91	= 136	18	18	18	112	= 166
22	51	= 107	15	28	28	66	= 137	18	34	34	81	= 167
22	70	= 109	6	15	28	91	= 140	7	18	34	112	= 171
35	51	= 110	15	15	45	66	= 141	18	18	55	81	= 172
51	51	= 112	6	6	66	66	= 144	7	7	81	81	= 176
35	51	= 113	6	28	45	66	= 145	7	34	55	81	= 177
35	70	= 115	6	6	45	91	= 148	7	7	55	112	= 181
22	70	= 116	15	15	28	91	= 149	18	18	34	112	= 182
35	70	= 118	1	15	45	91	= 152	1	18	55	112	= 186
22	70	= 119	6	28	28	91	= 153	7	34	34	112	= 187
12	92	= 121	6	15	15	120	= 156	7	18	18	148	= 191
35	70	= 122	6	15	45	91	= 157	7	18	55	112	= 192
22	92	= 124	6	6	28	120	= 160	7	7	34	148	= 196
51	51	= 125	1	28	66	66	= 161	1	34	81	81	= 197
22	92	= 127	1	15	28	120	= 164	1	18	34	148	= 201
35	70	= 128	1	28	45	91	= 165	1	34	55	112	= 202
35	35	= 127	28	45	45	45	= 163	34	55	55	55	= 199
35	70	= 129	15	15	45	91	= 166	18	18	55	112	= 203

DIAGRAM No. 4.

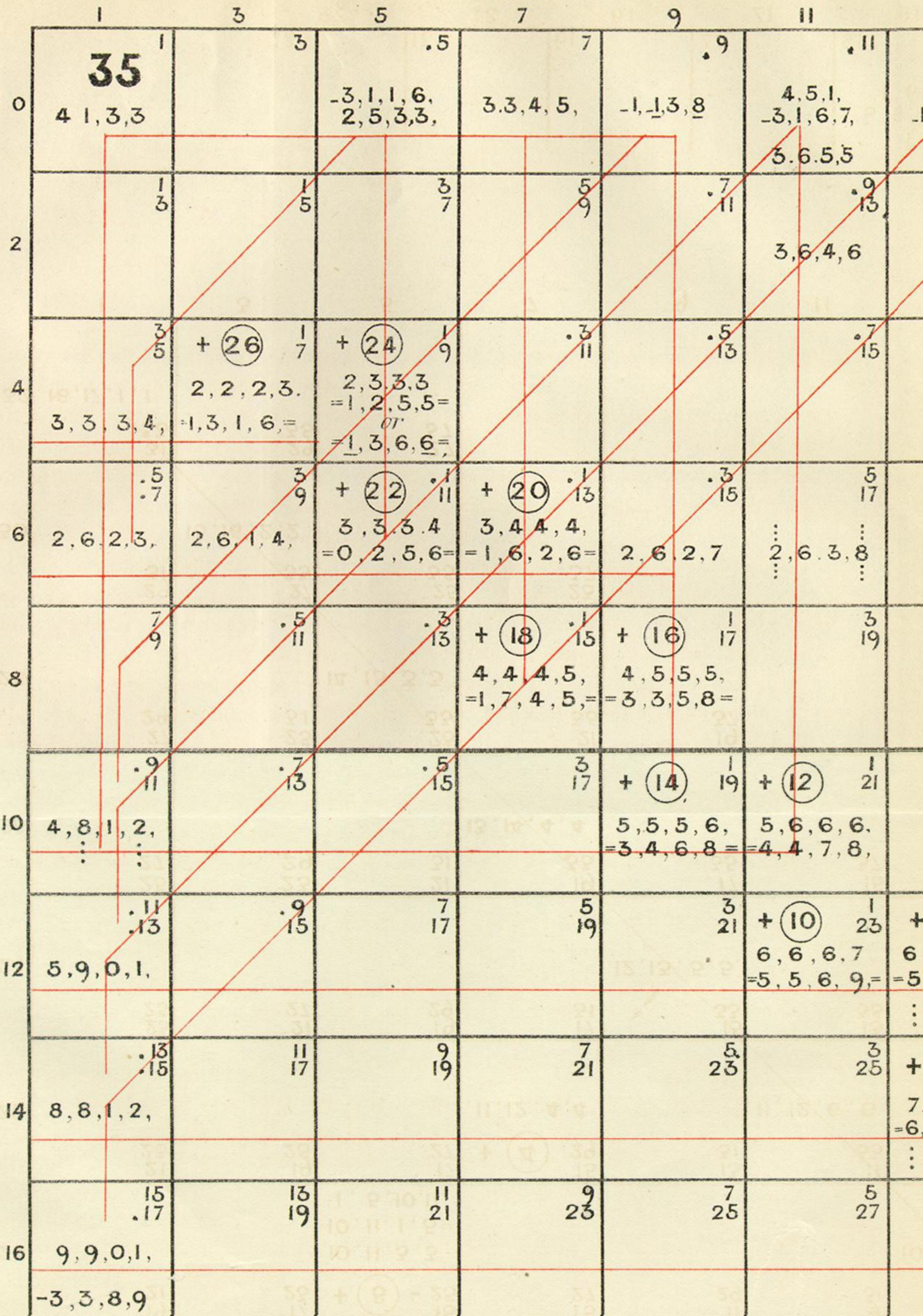
Natural numbers .	1	2	3	4	5	6	7	8	9	10	&c.
Trigonal numbers .	1	3	6	10	15	21	28	36	45	55	&c.
Squares	1	4	9	16	25	36	49	64	81	100	&c.
Pentagonal . . .	1	5	12	22	35	51	70	92	117	145	&c.
Hexagonal . . .	1	6	15	28	45	66	91	120	153	190	&c.
Heptagonal . . .	1	7	18	34	55	81	112	148	189	235	&c.
Octagonal , . .	1	8	21	40	65	96	133	176	225	280	&c.

EXPLANATORY REMARKS ON DIAGRAM No. 2.

The number in the first square is 35, a number of the form $4n+3$; the nearest division in whole numbers of 35 is 17, 18, and the nearest division of these is 8, 9, 9, 9, which are the roots of the $(n+1)$th term down the diagonal (in this case the 9th); these roots, when 8 and 9 are negative, give a difference of 1, and therefore the roots traverse from the place they occupy to the top corner, becoming there 0, 1, 17, 17, and also to the opposite corner, becoming there 1, 1, 16, 17. If the roots ascend in the diagonal towards 35, diminishing each root every step by one, they will require a numeral to be added, viz. 4, 6, 12, 16, &c., to make them equal to the number that should be found in the square they occupy. This may be easily done by the assistance of the 2 columns of $2b^2$ and $2a^2+2a$; a few of them are corrected and marked thus, $=0, 2, 5, 6=$ by way of example. The term in the gradation series next below 8, 9, 9, 9 has the roots 8, 8, 8, 9, but it requires 2 to be added, and when corrected is 7, 9, 8, 9. If the roots 8, 8, 8, 9 (diminished by 1 each) at every step be taken up the 2nd diagonal, they will require a numeral to be added, viz. 2, 6, 10, 14, &c., to make them equal to the required number; this also may be done in the same way as the other, whenever the correction produces 1 as a root; the roots of the 1st number may be obtained; thus in the 4th square of the diagonal the corrected roots are 1, 6, 2, 6, the sum of which is 15; but by making 1 negative, $\overset{3}{-1}, \overset{4}{2}, \overset{0}{6}, 6=13$, and lessening the roots by 3 each, they become $-4-133$, the sum of whose squares is 35, and the algebraic sum of the roots 1; the number $\frac{35-1}{2}$, that is 17, is composed of 3 trigonal numbers, viz. 10, 6, 1. All the squares in the line have a similar property, 12, 13, 5, 5; the 4th below to the left from 8, 9, 9, 9 may be carried up, diminishing the roots by 1 each every step, but adding as a numeral 4, 8, 12, &c., the roots will always (one or more) have 1 among them; but it may not always be easy to find out which; the following mode cannot fail: 8, 8, 8, 9 is 34 less than 8, 9, 9, 9, each term in the Gradation Series is less than the next above it by the sum of the roots of the next, minus 1; 34 will be added by adding 18, 12, and 4; 18 will be added by changing 8, 8 into 5, 11; 12 may be added by changing 8, 9 into 6, 11; the roots will then be 5, 6, 11, 11, and to these 4 will be added by changing 5, 6 into 4, 7; 4, 7, 11, 11, when squared (each of them), $=8, 9, 9, 9$ when squared and added together; but the sum of the roots of 4, 7, 11, 11 is 33, the index of the square in which 8, 9, 9, 9 is; it will therefore traverse upwards to the left and diminish each root by 8; the roots will be $-4-$ 133 as before.

Note.—Every square that is crossed by a red line, whether perpendicular, horizontal, or transverse, will have one or more forms of roots, the sum of the squares of which will be the number belonging to that square.

DIAGRAM No. 2.


1 3 5 7 9 11
0 2 4 6 8 10 12 14 16
1
35
4 1,3,3
3
.5
-3,1,1,6,
2,5,3,3,
7
3.3,4,5,
.9
-1,-1,3,8
.11
4.5.1,
-3,1,6,7,
3.6.5,5
1/3
1/5
3/7
5/9
.7/11
.9/13
3,6,4,6
3/5
3,3,3,4,
+ 26
1/7
2,2,2,3.
-1,3,1,6,=
+ 24
1/9
2,3,3,3
=1,2,5,5=
or
=1,3,6,6=,
.3/11
.5/13
.7/15
.5/.7
2,6,2,3,
3/9
2,6,1,4,
+ 22
.1/11
3,3,3,4
=0,2,5,6=
+ 20
.1/13
3,4,4,4,
=1,6,2,6=
.3/15
2,6,2,7
5/17
2,6,3,8
7/9
.5/11
.3/13
+ 18
.1/15
4,4,4,5,
=1,7,4,5,=
+ 16
1/17
4,5,5,5,
=3,3,5,8=
3/19
.9/11
4,8,1,2,
.7/13
.5/15
3/17
+ 14
1/19
5,5,5,6,
=3,4,6,8 =
+ 12
1/21
5,6,6,6,
=4,4,7,8,
.11/.13
5,9,0,1,
.9/15
7/17
5/19
3/21
+ 10
1/23
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=5,5,6,9,=
.13/.15
8,8,1,2,
11/17
9/19
7/21
5/23
3/25
15/.17
9,9,0,1,
-3,3,8,9
13/19
11/21
9/23
7/25
5/27

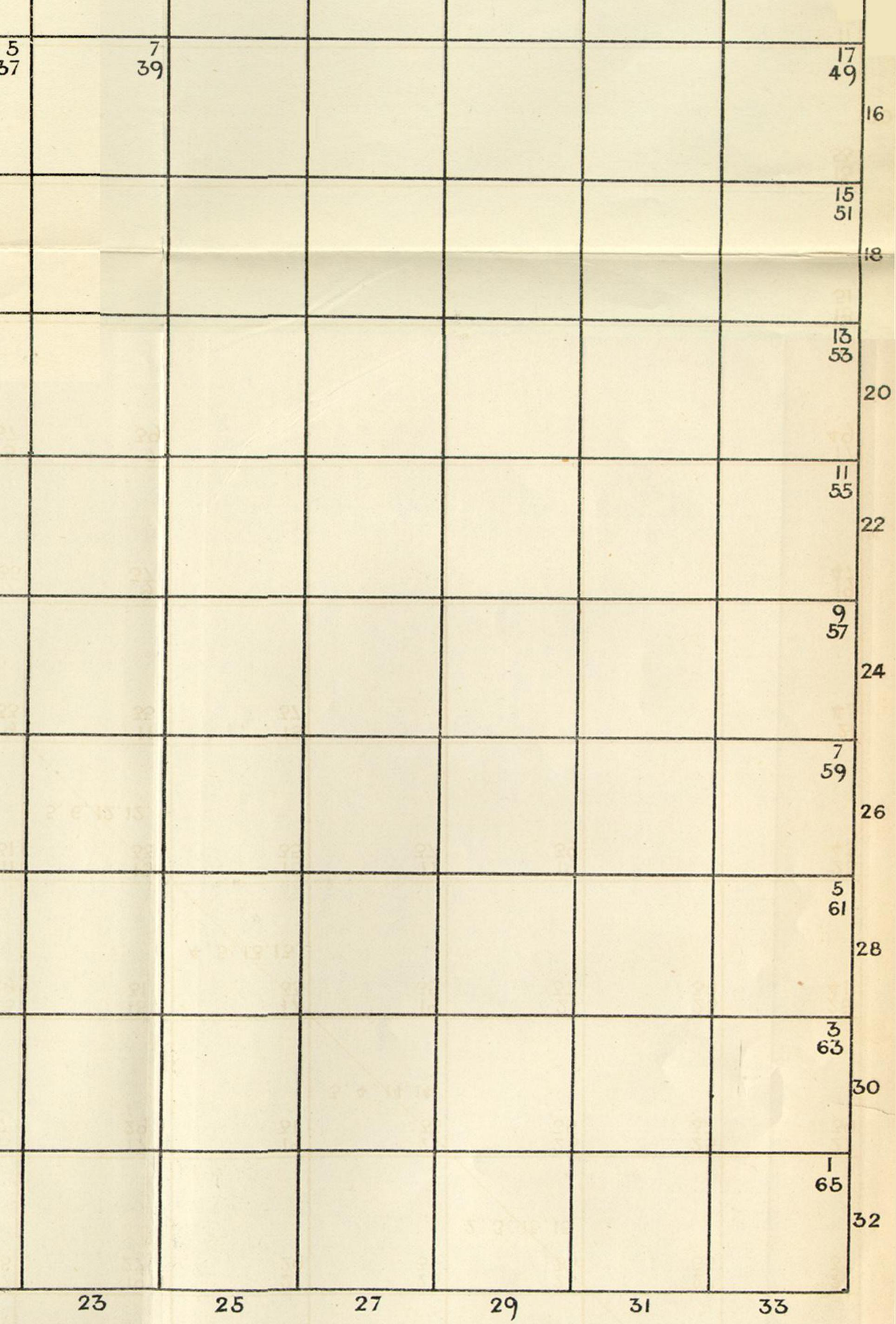
11	13	15	17	19	21	23
.11 4,5,1, -3,1,6,7, 3.6.5,5	.13 0,7,3. -1,1,6,9,	.15 0,9,1. -1 -1,8,9,	17	19	21	23
.9 13 3,6,4,6	.11 15	13 17	15 19	17 21	19 23	21 25
.7 15	9 17	11 19	13 21 1, 1, 8, 11,	15 23	17 25	19 27
5 17 2,6.3,8	7 19	9 21	11 23	13 25	15 27	17 29
3 19	5 21	7 23	9 25	11 27	13 29	15 31
+ 12 1 21 5,6,6,6. =4,4,7,8,	3 23	5 25	7 27	9 29	11 31	13 33 5, 6, 12, 12,
+ 10 1 23 6,6,6.7 =5,5,6,9,=	+ 8 1 25 6,7,7,7, =5,6,7,9,	3 27	5 29	7 31	9 33 6, 7, 11, 11,	11 35
3 25	+ 6 1 27 7,7,7,8 =6,8,6,9=	+ 4 1 29 7,8,8,8, =6,9,8,8,=	3 *as* 8+6 31 *equals* 14 6,9,8,8 *goes up and becomes* 8.9.1.1.	5 33 7,8,10,10	7 35	9 37
5 27	3 29	+ 2 1 31 8, 8, 8, 9, =7, 9, 8, 9,=	1 33 8, 9, 9, 9,	3 35	5 37	7 39

	23	25	27	29	31	33	
21	23	25	27	29	31	33 0.1, 17, 17	0
19 23	21 25	23 27	25 29	27 31	29 33 1.2, 16,16.	31 35	2
17 25	19 27	21 29	23 31	25 33 2, 3, 15, 15	27 35	29 37	4
15 27	17 29	19 31	21 33 3, 4, 14, 14,	23 35	25 37	27 39	6
13 29	15 31	17 33 4, 5, 13, 13,	19 35	21 37	23 39	25 41	8
11 31	13 33 5, 6, 12, 12,	15 35	17 37	19 39		23 43	10
9 33 1,	11 35	13 37				21 45	12
7 35	9 37					19 47	14
5 37	7 39					17 49	16

16	15 .17 9,9,0,1, -3,3,8,9	13 19	11 21	9 23	7 25	5 27
18	17 19	15 21	13 23	11 25	9 27	7 29
20	19 21	17 23	+ 8 - 15 25 10,11,3,3 =10,11,1,5= -1, 5,10,11,	13 27	11 29	9 31
22	21 23	19 25	17 27	+ 4 15 29 11,12,4,4	13 31	11 33 11,12,6,6
24	23 25	21 27	19 29	17 31	15 33 12,13,5,5,	13 35
26	25 27	23 29	21 31	19 33 13,14,4,4	17 35	15 37
28	27 29	25 31	23 33 14,15,3,3	21 35	19 37	
30	29 31	27 33 15,16,2,2	25 35	23 37		
32	31 33 16,17,1,1	29 35	27 37			
	1	3	5	7	9	11

10

	⋮ ⋮	⋮ ⋮	8. 9. 1. 1.				
5 27	3 29	+ 2 1 31 8, 8, 8, 9, =7, 9, 8, 9,=	1 33 8, 9, 9, 9,	3 35	5 37	7 39	
7 29	5 31	3 33 9, 10, 8, 8,	1 35	1 37			
9 31	7 33 10, 11, 7, 7	5 35	3 37				
11 33 11, 12, 6, 6	9 35	7 37					
13 35	11 37						
15 37							
11	13	15	17	19	21	23	



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